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SOME PROPERTIES OF EQUATIONS AND THE METHOD OF THE SMALL PARAMETER IN TWO-DIMENSIONAL SPATIAL PROBLEMS OF THE THEORY OF IDEAL PLASTICITY

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Interest in two-dimensional spatial problems of ideal plasticity theory is due, to a significant degree, to the fact that the majority of industrial operations (rolling, drawing, and immersion of tubes and rods) leads to the study of problems in a three-dimensional coordinate system in which the quantities to be determined depend on two coordinates. We assume that the components σ_{ij} (i, j = 1, 2, 3) of the stress tensor and the components v_i of the velocity vector depend on two variables q_1, q_2 in an orthogonal curvilinear coordinate system q_i ($\sigma_{13} = \sigma_{23} = v_3 = 0$). The Lamé coefficients $H_\ell^2 = \sum_{k=1}^{3}\left(\frac{\partial g_l}{\partial x_k}\right)^2$ are also functions of these coordinates: $H_\ell = H_\ell(q_1, q_2)$. Included in this class are axisymmetric problems (r, z, θ), problems in a spherical coordinate system (r, θ, φ), problems for bodies bounded by coordinate surfaces of degenerate "oblate" and "prolate" ellipsoids, toroidal coordinates, paraboloidal and bipolar coordinates of revolution [1, 2], and many others. The most completely investigated problems are the axisymmetric problems with a Tresca-type plasticity condition [3-5] and some special regimes. Most of the exact and approximate solutions have been obtained for total plasticity [6-10], when the problem becomes locally statically determinate and the system of equations in the stresses and velocities is of hyperbolic type.

The intensive use in recent years of anisotropic powderlike materials, as well as of materials having diverse yield limits in tension, compression, and shear, calls for an analysis of the equations under a more general yield condition. Such an analysis makes it possible not only to extend the class of exact analytical solutions, but also to develop a uniform method for obtaining sufficiently reliable approximate solutions in the event the formulation of exact solutions is not possible. Our aim in the present paper is to analyze some general properties of the equations for two-dimensional spatial problems of plasticity theory and to develop, based on these properties and also extremal properties of limiting loads for rigid-plastic bodies, a uniform method for solving such problems.

1. In the general case involving two-dimensional problems of an ideally rigid-plastic orthotropic body (coordinate axes coinciding with the axes of orthotropy) it is assumed that in the four-dimensional stress space there exists a nonconcave piecewise-smooth yield surface and that there is a valid associated plastic yield law

$$\varepsilon_{ii} = \mu_k \frac{\partial F_k}{\partial \sigma_{ii}}, \quad 2\varepsilon_{12} = \mu_k \frac{\partial F_k}{\partial \sigma_{12}} \tag{1.1}$$

(no summation on i), where

$$\begin{gathered} \mu_k = 0, \quad \text{if} \quad F_k < 0 \quad \text{or} \quad F_k = 0,\ dF_k < 0, \\ \mu_k > 0, \quad \text{if} \quad F_k = 0 \text{ and } dF_k = 0; \\ \varepsilon_{11} = \frac{1}{H_1} v_{1,1} + \frac{v_2}{H_1 H_2} H_{1,2}, \quad \varepsilon_{22} = \frac{1}{H_2} v_{2,2} + \frac{v_1}{H_1 H_2} H_{2,1}, \end{gathered} \tag{1.2}$$

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$$\varepsilon_{33} = \frac{v_1}{H_1H_3}H_{3,1} + \frac{v_2}{H_2H_3}H_{3,2},\quad 2\varepsilon_{12} = \frac{H_2}{H_1}\left(\frac{v_2}{H_2}\right)_{,1} + \frac{H_1}{H_2}\left(\frac{v_1}{H_1}\right)_{,2}. \tag{1.2}$$

The components of the stress tensor must satisfy the equilibrium equations

$$\begin{gathered}(H_2H_3\sigma_{11})_{,1} + (H_1H_3\sigma_{12})_{,2} + \sigma_{12}H_3H_{1,2} - \sigma_{22}H_3H_{2,1} - \sigma_{33}H_2H_{3,1} = 0,\\ (H_2H_3\sigma_{12})_{,1} + (H_1H_3\sigma_{22})_{,2} + \sigma_{12}H_3H_{2,1} - \sigma_{11}H_3H_{1,2} - \sigma_{33}H_1H_{3,2} = 0.\end{gathered} \tag{1.3}$$

For the singular regime $F_1 = F_2 = 0$, formed by the intersection of two smooth surfaces, the stress tensor components can be expressed in terms of two parameters and the problem becomes locally statically determinate. We introduce the following notation:

$$\begin{gathered}a_{ijk} = \partial F_k/\partial\sigma_{ij},\ \Delta_1 = a_{111}a_{332} - a_{112}a_{331},\\ \Delta_2 = a_{221}a_{332} - a_{222}a_{331},\ \Delta_3 = a_{121}a_{332} - a_{331}a_{122},\\ \Delta_4 = a_{111}a_{122} - a_{112}a_{121},\ \Delta_5 = a_{221}a_{122} - a_{222}a_{121}.\end{gathered}$$

Using the implicit function theorem [11], we can determine, depending on the sign of $\Delta = \Delta_3{}^2 - 4\Delta_1\Delta_2$, the type of system of equations from the expressions $\Delta > 0$, $\Delta = 0$, $\Delta < 0$, i.e., whether the system is hyperbolic, parabolic, or elliptic [12].

Similarly, expressing the μ_k in terms of the deformation rates ε_{33}, ε_{12}, we obtain, from the associated law (1.1), a system of equations in the rates:

$$\Delta_3\varepsilon_{11} = 2\Delta_1\varepsilon_{12} + \Delta_4\varepsilon_{33},\ \Delta_3\varepsilon_{22} = 2\Delta_2\varepsilon_{12} + \Delta_5\varepsilon_{33}. \tag{1.4}$$

The equations of the characteristics and the relations on them are determined by the expressions:

$$\frac{dq_2}{dq_1} = \lambda_m = \frac{H_1}{H_2}\left(\frac{-\Delta_3 \pm \sqrt{\Delta}}{2\Delta_2}\right)\quad (m = 1, 2), \tag{1.5}$$

$$\begin{gathered}\Delta_1H_1H_2H_3d\sigma_{11} - \lambda_m\Delta_2H_2^2H_3d\sigma_{12} + \{[\Delta_1H_1(H_2H_3)_{,1} + \lambda_m\Delta_2H_2H_3H_{1,2}]\sigma_{11} +\\ + [\Delta_1(H_1^2H_3)_{,2} - \lambda_m\Delta_2(H_2^2H_3)_{,1}]\sigma_{12} + [\lambda_m\Delta_2H_{3,2} - \Delta_1H_{3,1}]H_1H_2\sigma_{33} -\\ - [\Delta_1H_1H_2H_{2,1} + \lambda_m\Delta_2H_2(H_1H_3)_{,2}]\sigma_{22}\}dq_1 = 0,\\ H_1dv_1 + \lambda_mH_2dv_2 + \{H_1H_{1,2}H_2^{-1}v_2 + (\lambda_m)^2H_2H_{2,1}H_1^{-1}v_1 -\\ - \lambda_m[v_1H_{1,2} + v_2H_{2,1}] + [\Delta_4H_1^2 + \lambda_m^2\Delta_5H_1H_2]\Delta_3^{-1}[v_1H_{3,1}(H_1H_3)^{-1} +\\ + v_2H_{3,2}(H_2H_3)^{-1}]\}dq_1 = 0.\end{gathered}$$

An example of a singular regime is the condition for total plasticity [5]

$$F_1 = (\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 - 4k^2 = 0,\quad F_2 = \sigma_{33} - (\sigma_{11} + \sigma_{22})/2 \pm k = 0.$$

In this case,

$$\Delta_1 = -\Delta_2 = 2(\sigma_{11} - \sigma_{22}),\ \Delta_3 = 8\sigma_{12},\ \Delta = 16k^2 > 0,$$

i.e., the system of equations is hyperbolic and the characteristics of the different families are orthogonal.

For an incompressible material the singular regime always leads to a hyperbolic system of equations, the characteristics of the distinct families being mutually orthogonal. Actually, for a plastic incompressible material $\partial F_k/\partial\sigma_{ii} = 0$ $(k = 1, 2)$ or $a_{33k} = -a_{11k} - a_{22k}$, $\Delta_1 = -\Delta_2$, $\Delta \geq 0$. If $\Delta > 0$, then $\lambda_1\lambda_2 = -H_1{}^2H_2{}^{-2}$, i.e., the characteristics are mutually orthogonal.

In the case of multiple roots $\lambda_1 = \lambda_2$ the relations on a characteristic have the form

$$\frac{dq_1}{dq_2} = \lambda_1 = -\frac{\Delta_3H_1}{2\Delta_2H_2}, \tag{1.6}$$

$$\begin{gathered}H_2^2H_3d\sigma_{12} + [H_2(H_3H_1)_{,2}\sigma_{22} - H_2H_3H_{1,2}\sigma_{11} +\\ + \sigma_{12}(H_2^2H_3)_{,1} - H_1H_2H_{3,2}\sigma_{33}]dq_1 = 0,\\ H_1H_2H_3d\sigma_{11} + [H_1(H_2H_3)_{,1}\sigma_{11} - H_1H_3H_{2,1}\sigma_{22} +\end{gathered}$$

$$+\sigma_{12}(H_1^2H_3)_{,2}-H_1H_2H_{3,1}\sigma_{33}]\,dq_1=0, \tag{1.6}$$
$$dv_1+\{H_{1,2}H_2^{-1}v_2+H_1\Delta_4\Delta_3^{-1}[v_1H_{3,1}(H_1H_3)^{-1}+v_2H_{3,2}(H_2H_3)^{-1}]\}\,dq_1=0,$$
$$dv_2+\{H_{2,1}H_1^{-1}v_1+H_2\Delta_5\Delta_3^{-1}[v_1H_{3,1}(H_1H_3)^{-1}+v_2H_{3,2}(H_2H_3)^{-1}]\}\,dq_1=0.$$

Taking into account that the edges of the yield surface can be regarded as the result of an infinite variety of ways to approximate the initial yield surface, we see that the analysis carried out can be useful in constructing nonformal approximate solutions of the initial problem by going from a "nonsuitable" system to a more "suitable" one. For example, if the initial piecewise-smooth yield surface leads to an elliptic system of quasilinear equations whose solution cannot be found, the problem can be reduced to two amenable hyperbolic systems by introducing a manifold of approximating yield surfaces possessing the very weak constraint: $\Delta > 0$. If we take into account the fact that such a manifold contains both inscribed and circumscribed surfaces with respect to the initial yield surface, we can, in this way, construct in a uniform manner simultaneously upper and lower bounds to the initial loads.

2. Let us assume that the solution of the problem depends in a regular way on the small parameter δ. In accordance with the method of perturbations [13], we seek a solution in the form of a series in powers of the parameter δ:

$$\sigma_{ij}=\sum_{n=0}^{\infty}\delta^n\sigma_{ij}^{(n)},\quad v_i=\sum_{n=0}^{\infty}\delta^n v_i^{(n)},\quad \mu_k=\sum_{n=0}^{\infty}\delta^n\mu_k^{(n)}. \tag{2.1}$$

If we write the yield conditions in the form (2.1) and equate coefficients of identical powers of the parameter in the relations (1.1)-(1.3), we obtain a sequence of systems of equations for the approximations. For the singular regime

$$F_1=F_2=0 \tag{2.2}$$

the type of system of equations coincides precisely with the type of system of equations of the zeroth approximation. For a nonelliptic system of equations the characteristics for an arbitrary approximation coincide with the characteristics of the zeroth approximation. Indeed, from the yield condition (2.2) and the law (1.1), we obtain

$$a_{ijk}^{(0)}\sigma_{ij}^{(n)}+f_k^{(n)}=0\quad(k=1,2); \tag{2.3}$$
$$\varepsilon_{ii}^{(n)}=\mu_k^{(n)}a_{iik}^{(0)}+f_{ii}^{(n)},\quad 2\varepsilon_{12}^{(n)}=\mu_k^{(n)}a_{12k}^{(0)}+f_{12}^{(n)} \tag{2.4}$$

(no summation on i). Here $f_k^{(n)}$, $f_{ij}^{(n)}$ depend on the previous approximations and, consequently, are known functions. Substituting the relations (2.3) into the equations (1.3), we note that the type of system obtained coincides exactly with the type of system for the zeroth approximation. The system of equations for the rates is analogous to the system (1.4). In the case of an equation of hyperbolic type ($\Delta^{(0)} > 0$) the characteristics coincide with the expressions (1.5), and the relations on them, for an arbitrary approximation, are determined by the following equations:

$$\Delta_1^{(0)}H_1H_2H_3d\sigma_{11}^{(n)}-\Delta_2\lambda_m^{(0)}H_2^2H_3d\sigma_{12}^{(n)}+\{[\Delta_1^{(0)}H_1(H_2H_3)_{,1}+ \tag{2.5}$$
$$+\lambda_m^{(0)}\Delta_2^{(0)}H_2H_3H_{1,2}]\sigma_{11}^{(n)}+[\Delta_1^{(0)}(H_1^2H_3)_{,2}-\lambda_m^{(0)}\Delta_2^{(0)}(H_2^2H_3)_{,1}]\sigma_{12}^{(n)}-$$
$$-[\Delta_1^{(0)}H_1H_3H_{2,1}+\lambda_m^{(0)}\Delta_2^{(0)}(H_1H_3)_{,2}]\sigma_{22}^{(n)}+H_1H_2[\lambda_m^{(0)}\Delta_2^{(0)}H_{3,2}-\Delta_1^{(0)}H_{3,1}]\times$$
$$\times\sigma_{33}^{(n)}-H_1H_2H_3\Delta_1^{(0)}[(\Delta_2^{(0)}/\Delta_1^{(0)})_{,1}\sigma_{22}^{(n)}+(\Delta_3^{(0)}/\Delta_1^{(0)})_{,1}\sigma_{12}^{(n)}+$$
$$+((a_{332}^{(0)}f_1^{(n)}-a_{331}^{(0)}f_2^{(n)})/\Delta_1^{(0)})_{,1}]\}\,dq_1=0,$$
$$H_1dv_1^{(n)}+\lambda_m^{(0)}H_2dv_2^{(n)}+\{H_1H_2^{-1}H_{1,2}v_2^{(n)}+(\lambda_m^{(0)})^2H_2H_1^{-1}H_{2,1}v_1^{(n)}-$$
$$-\lambda_m^{(0)}(v_1^{(n)}H_{1,2}+v_2^{(n)}H_{2,1})+(\Delta_4^{(0)}H_1^2-(\lambda_m^{(0)})^2\Delta_5H_1H_2)(v_1^{(n)}H_{3,1}(H_1H_3)^{-1}+$$
$$+v_2^{(n)}H_{3,2}(H_2H_3)^{-1})/\Delta_3^{(0)}-f_{11}^{(n)}-\lambda_m^{(0)}H_2H_1^{-1}f_{12}^{(n)}-(\lambda_m^{(0)})^2H_2^2H_1^{-2}f_{22}^{(n)}-$$
$$-(\Delta_4^{(0)}+\Delta_5^{(0)}(\lambda_m^{(0)})^2H_2H_1^{-1})f_{33}^{(n)}/\Delta_3^{(0)}\}\,dq_1=0.$$

The property of conservation of characteristics can be used effectively in the construction of a numerical algorithm since the necessity of rebuilding the field of characteristics for each approximation is eliminated. This method is especially effective when a piecewiselinear yield surface is used. In this case, $\alpha_{ijk}^{(0)}$ = const, $f_k^{(n)} = f_{ij}^{(n)} = 0$, and the expressions are preserved on the known characteristics, coinciding exactly with the relations for the zeroth approximation. In the general case of piecewise-smooth surfaces and the singular regime, $f_k^{(n)}$, $f_{ij}^{(n)} \neq 0$, use of the method becomes difficult because of the need to maintain all the previous approximations in the computer memory. Therefore the following procedure is possible. Having obtained the zeroth approximation, and having subsequently constructed the field of characteristics with the aid of the relations on them, we construct the field of stresses and rates for the first approximation; next, taking into account the first approximation, we obtain the field of stresses $\sigma_{ij} = \sigma_{ij}^{(0)} + \delta\sigma_{ij}^{(1)}$ and rates $v_i = v_i^{(0)} + \delta v_i^{(1)}$; this field is taken as the initial zeroth approximation, the characteristics are corrected, and the procedure continued until the required accuracy is attained.

3. In the general case of a regular regime when the solution of the problem becomes substantially more complicated, the problem ceases to be locally statically determinate [14, 15]. With the exception of certain special regimes in which the problem becomes kinematically determinate, there is a need for a joint solution of the system in terms of rates and stresses, and the resulting system of equations does not have (with a certain exception) real characteristics [15]. For an approximate solution in this case we can apply the following method. Let F = 0 be a smooth regime corresponding to the initial problem.

We seek a solution to a "fictitious" problem, having the boundary conditions of the initial problem, with a yield surface $F_1 = F_2 = 0$, where the F_k (k = 1, 2) possess properties intrinsic to the initial surface: a) the F_k are smooth and nonconcave; b) if the initial plastic material is incompressible, i.e., if $\partial F/\partial\sigma_{ii} = 0$, then the regimes F_k have the very same property: $\partial F_k/\partial\sigma_{ii} = 0$. We also select the surfaces F_k such that the σ_{ij}, satisfying the condition $F_1 = F_2 = 0$, automatically satisfy the condition F = 0. This can always be done in an infinite number of ways. For example, in using the von Mises criterion

$$F = (\sigma_{11} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2 + (\sigma_{22} - \sigma_{33})^2 + 6\sigma_{12}^2 - 6k^2 = 0$$

we have the following variants:

$$\begin{gathered} F_1^{\alpha_1\gamma} = \sigma_{33} - \alpha_1\sigma_{11} - \sigma_{22}(1 - \alpha_1) + \gamma k, \quad F_2^{\alpha_1\gamma} = 6\sigma_{12}^2 + (\sigma_{11} - \sigma_{22})^2 \times \\ \times [1 - 2(1 - \alpha_1)^2] - 4(1 - \alpha_1)(\sigma_{11} - \sigma_{22})\gamma k + 2k^2(\gamma^2 - 3). \end{gathered} \tag{3.1}$$

Depending on how we choose (α_1, γ), we obtain a whole spectrum of "fictitious" problems. The field of stresses obtained for a "fictitious" problem is statically permissible for the initial problem since it satisfies, by construction, the boundary conditions for the stresses, the equations of equilibrium, and the yield conditions. The field of rates for the "fictitious" problem will be kinematically feasible for the initial problem since it satisfies the boundary conditions for the rates and, in the case of an incompressible material, the condition of incompressibility. The fields of rates and stresses for the "fictitious" problem satisfy all the conditions, but they violate the associated law of plastic flow of the initial problem. Using extremal principles in the theory of an ideal rigid—plastic body [17], we make it possible through their use to obtain lower and upper estimates for the level of limiting external loads. A wide spectrum of possibilities of a single type allows us to select a best estimate.

The best variants with respect to the stresses and rates will be, respectively, those which give a maximum estimate of the load level with respect to the stresses and a minimum with respect to the rates. It is possible to have cases in which the best estimates yield solutions constructible from a combination of these variants in various regions of the problem studied; i.e., when the parameters α_1 and γ in the conditions (3.1) vary discretely or continuously in the various regions. In the case of an incompressible body, the type of the equations for the "fictitious" variants is hyperbolic; this allows us to use the method of characteristics and to algorithmatize the selection of the variants. In solving "fictitious" problems it may turn out to be useful to apply the method of expansion with respect to a small parameter.

We remark that in the study of singular regimes we need to have $\mu_k > 0$; this agrees with the requirement of positiveness of energy dissipation in plastic regions. In solving "fictitious" problems the condition $\mu_k > 0$ may be violated. Despite this, the rate and stress fields obtained will be statically admissible and kinematically possible for the initial problem and the requirement $\mu_k > 0$ may be waived.

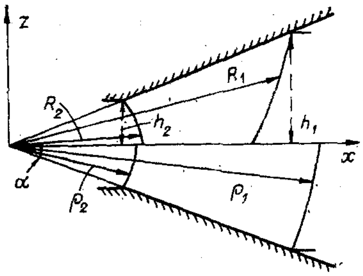

Fig. 1

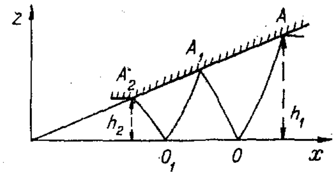

Fig. 2

The method of constructing two-sided estimates of the limiting loads was widely used in the analysis of industrial problems involving plasticity [18-20]; however, the selection of kinematically possible rate fields and statically admissible stress fields was stipulated to a significant degree by the subjective qualities of the investigator. Here, as an example, we illustrate the general approach proposed by considering a problem dealing with the drawing and extrusion of rods (Fig. 1) subject to the von Mises yield condition. We assume that in contact with the die in the spherical r, θ, φ coordinate system the law of constancy of friction, $\sigma_{12} = \mu k$ is valid. We introduce a "fictitious" problem with the yield condition (3.1) with $\gamma = \sqrt{3}(1 - \alpha_1)$; we seek its solution in the form of an expansion in terms of the small parameter μ. Then in the zeroth approximation in the conical portion of the die we assume a radial flow of the material:

$$v_1^{(0)} = 2V_k h_k^2 r^{-2}(1-\cos\alpha)^{-1},$$
$$\sigma_{11}^{(0)} = -2\sigma_s \ln r + \mathrm{const}, \quad \sigma_{22}^{(0)} = \sigma_{33}^{(0)} = \sigma_{11}^{(0)} - \sigma_s, \quad \sigma_{12}^{(0)} = 0, \quad \sigma_s = \sqrt{3}k,$$

where V_k (k = 1, 2) is the rate at the entrance and at the exit of the die. The characteristics of the system (Fig. 2) are the logarithmic spirals $r = \mathrm{const}\exp(\pm\theta)$. In the regions AOA_1, OA_1O_1, $A_1O_1A_2$ we solve successively mixed boundary-value problems with the aid of the relations (2.5) on the characteristics. The boundary-value problems have the form $\sigma_{12}^{(n)} = \sigma_{11}^{(n)} = 0$ on OA; $\sigma_{12}^{(1)} = k$, $\sigma_{12}^{(n)} = 0$ on AA_2 (n > 1); $\sigma_{12}^{(n)} = 0$ on OO_1. Calculation of the rates is effected in the reverse order; moreover, the normal component of the rate vector is given on the curves A_2O_1, A_2A, and O_1O. The use of numerical procedures of the same type allows us to obtain a sufficient number of approximations. To obtain a rough estimate in closed form in the conical portion of the die, we assume radial flow of the material. Then, taking the first approximation into account we have

$$v_1 = Br^{-2}\left[1 + 4\sqrt{3}\mu \operatorname{ctg}\frac{\alpha}{2} \ln\cos\frac{\theta}{2}\right], \tag{3.2}$$
$$\sigma_{11} = A - 2\sigma_s \ln r - \mu\left[k \operatorname{ctg}\frac{\alpha}{2} \ln r + 6k \operatorname{ctg}\frac{\alpha}{2} \ln\cos\frac{\theta}{2}\right],$$
$$\sigma_{12} = \mu k \operatorname{tg}\frac{\theta}{2} \operatorname{ctg}\frac{\alpha}{2},$$

where A and B are constants.

From the condition of incompressibility of the medium it follows that

$$2\int_0^\alpha r^2 v_1(\theta)\sin\theta\, d\theta = V_k h_k^2,$$
$$B = V_k h_k^2\left[2(1-\cos\alpha) - 8\sqrt{3}\mu \operatorname{ctg}\frac{\alpha}{2}\left(\cos^2\frac{\alpha}{2}\ln\cos\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\right)\right]^{-1}.$$

The boundaries separating the rigid and the plastic regions, i.e., the slippage surfaces, may, with the first approximation taken into account, be determined in the region θ ≥ 0 (see Fig. 1) in the form

$$r^{-1}\frac{dr}{d\theta} = \operatorname{tg}\varphi_1; \ -\operatorname{ctg}\varphi_1, \ \operatorname{ctg}2\varphi_1 = \frac{\sigma_{22}-\sigma_{11}}{2\sigma_{12}}.$$

Here, from the entrance side

$$r = R_1(\theta) = \frac{h_1}{\sin\alpha}\exp(\theta-\alpha)\left[\frac{\cos(\theta/2)}{\sin(\alpha/2)}\right]^{\frac{\mu\operatorname{ctg}\frac{\alpha}{2}}{\sqrt{3}}}, \tag{3.3}$$

from the exit side

$$r = R_2(\theta) = h_2 h_1^{-1}\exp 2(\alpha-\theta).$$

The minimum value of the drawn portion, $\lambda = h_1 h_2^{-1}$, for which our solution has meaning, is determined from the condition of the absence of a "rigid plug" and has, in accordance with the relation (3.3), the form $\lambda_{min} = \exp 2\alpha$. Integrating along the corresponding boundaries, subject to the condition of no countertension (counterpressure), we obtain (to within terms of second order of smallness) a lower bound to the operational stress:

$$P^* \approx 2\pi\sigma_s h_k^2\left(1+\frac{\mu}{2\sqrt{3}}\operatorname{ctg}\frac{\alpha}{2}\right)\ln\lambda.$$

To determine an upper bound to this stress we draw the rigid-plastic boundaries in such a way that continuity of the normal component of the rate vector [17] is maintained. This condition is satisfied along the geometrically similar boundaries

$$r = \rho_k(\theta) = \frac{h_k}{\sin\theta}\left[\frac{(1-\cos\theta) - 8\sqrt{3}\operatorname{ctg}\frac{\alpha}{2}\left(\cos^2\frac{\theta}{2}\ln\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)}{(1-\cos\alpha) - 8\sqrt{3}\operatorname{ctg}\frac{\alpha}{2}\left(\cos^2\frac{\alpha}{2}\ln\cos\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\right)}\right]^{1/2}.$$

An upper bound to the stress is determined from the condition [18]

$$P_k^{**}V_k = N_1 + N_2 + N_3,$$

where N_1, N_2, and N_3 are, respectively, the magnitudes of the plastic deformation, the frictional forces in the die, and the shearing forces on the surfaces of discontinuity of the rates, which according to [16-18] have the form

$$N_1 = 2\pi k\int_0^{\alpha} r^2\ln\lambda\left[12v_1^2 + (v_{1,2})^2\right]^{1/2}\sin\theta\, d\theta \approx 2\pi\sigma_s V_k h_k^2\ln\lambda,$$

$$N_2 = 2\pi\mu k r^2\sin\alpha\ln\lambda v_1(\alpha) \approx \frac{\pi\mu\sigma_s}{\sqrt{3}}V_k h_k^2\ln\lambda\operatorname{ctg}\frac{\alpha}{2} + \mu^2\Phi_1(\alpha),$$

$$N_3 = 4\pi k V_1\int_0^{\alpha}\left[\rho_1^2 + (\rho_{1,2})^2\right]^{1/2}\sin^2\theta\, d\theta \approx \frac{\pi\sigma_s V_k h_k^2}{\sqrt{3}\,(1-\cos\alpha)} \times$$

$$\times\left(\alpha - 3\sin\alpha + 4\operatorname{tg}\frac{\alpha}{2}\right) + \mu\Phi_2(\alpha).$$

Noting that $\Phi_1(\alpha)$, $\Phi_2(\alpha) < 0$, we obtain a lower, and hence a more precise, upper bound compared to those given in [16-18]. Taking into account the strength of the friction forces in the calibrating portion of our instrument, we obtain finally an approximate estimate of the drawing stress:

$$p^* \approx 2\sigma_s\left[1+\frac{\mu}{2\sqrt{3}}\operatorname{ctg}\frac{\alpha}{2}\right]\ln\lambda,$$

$$p^{**} \approx p^* + \frac{\sigma_s\left|\alpha - 3\sin\alpha + 4\operatorname{tg}^2\frac{\alpha}{2}\right|}{\sqrt{3}\,(1-\cos\alpha)} + \mu\Phi_2(\alpha), \quad p = P/(\pi h^2).$$

Thus, our approach enables us to obtain a two-sided estimate of the operational stresses and, consequently, to estimate the error in the calculations. A comparison of the upper and lower bounds among themselves and also with known solutions [18-20] shows their accuracy to be satisfactory.

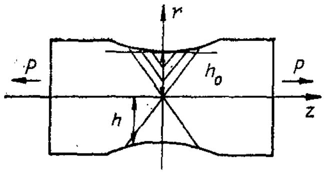

Fig. 3

As a second example of the construction of statically admissible estimates, we consider the stretching of a rod of circular cross section, weakened by an axisymmetric notch (Fig. 3), subject to the von Mises yield condition, and as the "fictitious" variants we consider the singular regimes (3.1). We assume that the notch is described by the equation

$$r = R_0 + R_1(\delta, z), \; R_1(0, z) = 0. \tag{3.4}$$

We seek a solution of the "fictitious" problem in the form of an expansion in the small parameter δ. Owing to symmetry under stretching, it is sufficient to consider the domain $r \geq 0$. Taking account of the fact that

$$\sigma_{11}^{(0)} = \sigma_{12}^{(0)} = \sigma_{33}^{(0)} = 0, \quad \sigma_{22}^{(0)} = \sqrt{3}k,$$

we need to put $\gamma = \sqrt{3}\alpha_1$, which limits the number of variants of the "fictitious" problem. With regard to the fact that $\Delta_1^{(0)} = -\Delta_2^{(0)}$, $\Delta_3^{(0)} = 0$, the equations of the characteristics and the relations on the characteristics for the first approximation may be written thus:

$$\frac{dz}{dr} = \lambda_m^{(0)} = \pm 1, \quad d\sigma_{11}^{(1)} + \lambda_m^{(0)}\left(d\sigma_{11}^{(1)} + r^{-1}\sigma_{12}^{(1)}dr\right) = 0. \tag{3.5}$$

The boundary condition on the unperturbed contour for the first approximation has the form

$$\sigma_{11}^{(1)} = 0, \quad \sigma_{(12)}^{(1)} = \sigma_{22}^{(0)} \frac{\partial}{\partial \delta} \frac{\partial R_1}{\partial z}\bigg|_{\delta=0} \quad \text{for} \quad r = R_0. \tag{3.6}$$

The solution for the first approximation can be realized numerically with the aid of finite differences. In the domain $0 \leq r \leq R_0$ we solve Cauchy problems with the relations (3.5) on the characteristics and with the initial conditions (3.6). We note, with regard to the first approximation, that these variants give identical estimates; this is not the case for calculation of the following approximations.

We consider now a problem concerning the pressing-in of a ring-shaped stamp into a half-space of an ideal rigid-plastic material obeying the von Mises plasticity condition. Introducing a fictitious problem in accordance with the relations (3.1), we write the stresses in the parametric form

$$\left.\begin{matrix}\sigma_{11}\\ \sigma_{22}\end{matrix}\right\} = p \pm BD_1^{-1}\cos 2\varphi \mp \gamma k(1-\alpha_1)D_1^{-2},$$

$$\sigma_{33} = p - B(1-2\alpha_1)D_1^{-1} - 3\gamma k D_1^{-2}, \quad \sigma_{12} = 3^{-1/2}B\sin 2\varphi,$$

$$B = 3^{1/2}k\left(1-\gamma^2 D_1^{-2}\right), \quad p = (\sigma_{11}+\sigma_{22})/2, \quad D_1^2 = 4\left(1-\alpha_1+\alpha_1^2\right), \; |\gamma| < D_1.$$

The characteristics and relations on the characteristics in the (r, z, θ) cylindrical coordinate system are:

$$\lambda_{1,2} = \left(\psi \pm \sqrt{1+\psi^2}\right), \quad \psi = \sqrt{3}D_1^{-1}\,\mathrm{tg}\,2\varphi,$$

$$dp \pm 2\cdot 3^{-1/2}B\sqrt{1+\psi^2}\cos 2\varphi d\varphi - \left[2B\left(D_1^2\cos 2\varphi\right)^{-1}\cos 4\varphi \mp \right.$$

$$\left. \mp 3^{-1/2}B\sqrt{1+\psi^2}\sin 2\varphi - 3\gamma k D_1^{-2}\right]dr = 0.$$

To obtain the solution in closed form we use the method of the small parameter, taking $\varepsilon = 1 - R_1R_2^{-1} < 1$ (R_1 and R_2 are the inner and outer radii of the ring-shaped stamp). For the first approximations we then obtain

$$dp^{(0)} \pm 2\cdot 3^{-1/2}\sqrt{1+\psi_0^2}\cos 2\varphi^{(0)}d\varphi^{(0)} = 0, \quad \psi_0 = \sqrt{3}D_1^{-1}\,\mathrm{tg}\,2\varphi^{(0)},$$

$$dp^{(1)} \pm 2\cdot 3^{-1/2}\sqrt{1+\psi_0^2}\cos 2\varphi^{(0)}d\varphi^{(1)} \mp \frac{B\left(D_1^2-3\right)\sin 2\varphi^{(0)}}{\sqrt{3}\,D_1^2\sqrt{1+\psi_0^2}}\,d\varphi^{(0)} -$$

$$-\left[\frac{B\cos 4\varphi^{(0)}}{D_1\cos 2\varphi^{(0)}} \mp \frac{B}{\sqrt{3}}\sqrt{1+\psi_0^2}\sin 2\varphi^{(0)} - \frac{3\gamma k}{D_1^2}\right]\frac{dr}{(R_2-R_1)} = 0,$$

and the boundary conditions are $\sigma_{12} = 0$, $\forall r$, $z = 0$; $\sigma_{22} = 0$, $z = 0$, $r \leq R_1$, $r \geq R_2$. The general pressing-in force is given by the expression

$$P = 2\pi\int_0^{R_2-R_1}\sigma_{22}(R_1+q_1)\,dq_1,$$

$$\sigma_{22} = p^{(0)} + \frac{(R_2-R_1)}{R_2}p^{(1)} - \frac{B}{D_1} + \frac{\gamma k(1-2\alpha_1)}{D_1^2}.$$

We obtain the parameters α_1 and γ from the condition of maximum of the force P. In this case, the solution may be expressed in terms of elliptic integrals. If for simplicity in our calculations we take $\alpha_1 = 1/2$, then

$$P = \pi\left[\frac{\gamma k(R_2-R_1)}{3R_2}\left(10R_2^2 - R_1R_2 + R_1^2\right) - \frac{2B\left(R_2^2-R_1^2\right)}{\sqrt{3}}\left(2+\pi+\frac{R_1}{R_2}\right)\right].$$

Taking note of the fact that $dP/d\gamma = 0$, $d^2P/d\gamma^2 < 0$, we have

$$\gamma = -\frac{10R_2^2 - R_1R_2 + R_1^2}{[(2+\pi)R_2 + R_1](R_2+R_1)}.$$

For a solid stamp ($R_1 = 0$)

$$\gamma = -1/6, \quad P = -(2+\pi+5/9)k\pi R_2 \approx -5{,}7k\pi R_2^2.$$

The force found numerically under the condition of total plasticity [6] is equal to $P = -5.8k\pi R_2{}^2$. Thus, our analytical solution yields an error of at most 2%. We can even find an upper bound to the force by constructing the rate field v_i with the aid of the relations on the characteristics:

$$dv_1 + \lambda_k dv_2 - v_1\left[\alpha_1 + (1+\alpha_1)\lambda_k^2\right]d\ln r = 0, \quad k = 1, 2.$$

Thus, we have proposed methods for solving systems of equations for two-dimensional spatial problems of the theory of an ideal rigid—plastic body in the case of singular regimes with a general piecewise-smooth yield surface; we have also proposed a method for estimating the admissible level of external forces for a smooth yield surface or for regular regimes based on the introduction of a "fictitious" problem and extremal properties for the multiplier of the external force level for an ideal rigid—plastic body.

Our approach may also find application in calculations for spatial structures of physical or structural materials, anisotropic or powderlike, and in many industrial problems involving the machining of metals.

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A CLASS OF COMPOSITE LOADS FOR AN INELASTIC MATERIAL

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In solid mechanics a considerable role is played by flow, which in a certain sense is the simplest form. In hydrodynamics this relates to Couette flow between parallel plates and coaxial cylinders [1], in solid mechanics it relates to deformation of thin-walled tubular specimens [2], and in the mechanics of loose materials it relates to uniform shear of the material [3]. Construction of sufficiently general phenomenological models assumes an experimental study of different loading paths, including composite loading paths when the stress tensor axes are turned relative to the volume of the material. Composite loading of metals, rocks, and other solids may be realized by a combination of internal pressure, torsion, and tension for tubular specimens. However, for a broad class of materials this classic procedure is either markedly complicated (e.g., for soils [4]), or it is generally inapplicable. It is of interest to find a class of composite loads which on one hand might relate to the simplest, and on the other might be used in order to test loose, viscoelastoplastic, and other similar materials.

1. As is well known, a uniform stress–strained state is the simplest. Let a material in the fixed direction be subjected to uniform tensile deformation $\Delta\varepsilon_1 = k\Delta t$, and in the orthogonal direction to compressive deformation so that the volume is unchanged; $\Delta\varepsilon_2 = -k\Delta t$. Then after time Δt the same uniform deformation is accomplished in new fixed directions turned relative to the previous directions by angle $-\Omega\Delta t$, etc. Deformation is planar, Ω and k are positive constants.

In order to derive equations, we consider a discrete sequence of these uniform loadings. Let $Ox_1'x_2'$ be the initial Cartesian coordinate system, and β the angle between the tensile direction Ox_1 and axis Ox_1' (Fig. 1). On coordinate Ox_1x_2 the vector for increment

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